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METHOD FOR SOLUTION OF SOME NON-LINEAR BOUNDARY VALUE PROBLEMS OF A NON-STATIONARY DIFFUSION-CONTROLLED (THERMAL) BOUNDARY LAYER

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Abstract—An exact analytical method is suggested for the solution of a wide class of unsteady state boundary layer problems which describe the processes of convective mass and heat transfer of particles in an ideal, and droplets (bubbles) in a viscous incompressible, fluid. The procedure proves also adequate in analyzing transfer processes occurring in the neighbourhood of a liquid–fluid type interface (e.g. in liquid films). The method is based on incorporation of four new independent variables specified by a set of 1st-order partial differential equations which allows a general solution of the linear problem subject to arbitrary initial and boundary conditions. In addition, it affords the possibility of obtaining a wide class of non-trivial (and non-self-similar) solutions of respective non-linear boundary value problems in the case when the diffusion coefficient (thermal diffusivity) is a function of concentration (temperature).

A general solution is obtained for a similar linear unsteady state problem connected with the 1st-order bulk chemical reaction occurring in the fluid. The process of convective diffusion to a particle at an arbitrary rate of chemical reaction proceeding on its surface is studied.

It is shown that solution to the unsteady state boundary layer equation can have a strong discontinuity which displaces at terminal speed.

Extension of the method to analogous problems of unsteady state mass and heat transfer in a compressible fluid is indicated.

The method suggested is used for the solution of a number of specific problems. The case of linear and hyperbolic dependence of the diffusion coefficient upon concentration is studied in detail.

NOMENCLATURE

A_α, A_β,	constants occurring in (41);
$\mathbf{A}$,	differential operator defined in (9);
a,	$= Y(0)$, constant occurring in (25);
a_e,	characteristic particle dimension;
B,	constant occurring in (41);
$B(x, p, q)$,	incomplete beta-function;
b,	$= Y(\infty)$, constant occurring in (25);
C,	limited concentration;
c_*,	dimensional reagent concentration in liquid flow;
c, c_α, c_β,	non-dimensional reagent concentration in liquid flow;
c_s,	surface concentration;
c_0,	characteristic concentration;
D_*,	$= D\sigma(c)$, diffusion coefficient;
D_s,	surface diffusion coefficient;
E,	function defined in (36);
$\mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\lambda$,	direction vectors of the curvilinear co-ordinate system;
F_*,	dimensional surface reaction rate;
f,	function occurring in (4);
G,	$= g_{\xi\xi} g_{\eta\eta} g_{\lambda\lambda}$, first invariant of metric tensor;
$g_{\xi\xi}, g_{\eta\eta}, g_{\lambda\lambda}$,	metric tensor components;
I, I_α, I_β,	non-dimensional total diffusional flux;
$I^0, I^0_\alpha, I^0_\beta$,	non-dimensional total diffusional flux corresponding to a linear problem at $\sigma = 1$;
I_m,	non-dimensional total diffusional flux to the *m*th droplet (particle) of a chain;
J,	function defining local diffusional flux (24);
j, j_α, j_β,	non-dimensional local diffusional flux;
$j^0, j^0_\alpha, j^0_\beta$,	non-dimensional local diffusional flux corresponding to the linear problem at $\sigma = 1$;
k, k',	non-dimensional and dimensional reaction rate constant, respectively;
$\mathbf{L}$,	differential operator defined in (5);
l,	non-dimensional distance between droplets (particles);
m,	ordinal number of a droplet (particle);
n,	exponent;
P,	$= P(Q)$, function defined in (35);
Pe,	$= a_e U_* D^{-1}$, Peclet number;
q,	source;
R,	function defined in (32);
R_s,	surface reaction rate;
r,	radial coordinate;
S,	particle surface area;
Sh,	mean Sherwood number;
T,	function defined in (3);
t,	non-dimensional time;
U_0,	flow velocity at infinity;
U,	$= U(t)$, function defined in (46);
U_*,	characteristic velocity of a particle;
$\mathbf{v}$,	fluid velocity;

v_ξ, v_η,	fluid velocity components;
V,	function occurring in (80);
x,	variable occurring in (25);
Y,	$= Y(x)$, solution of the problem (25);
Z, Z_f, Z_τ,	arbitrary functions depending upon the variable ω alone;
z,	$= z(\mu)$, solution of the auxiliary problem (31);
H,	integral operator defined in (39);
h,	$= h(\cos\theta)$, initial concentration distribution outside of the sphere.

Greek symbols

α,	shear factor;
Γ,	Henry constant;
γ,	coefficient occurring in (46);
Δ,	Laplace operator;
δ,	coefficient occurring in (72);
ε,	$= Pe^{-1/2}$, small parameter;
ζ,	new variable (4);
η,	second (longitudinal) curvilinear coordinate;
θ,	polar angle;
κ,	parameter occurring in (27) and (33);
Λ,	differential operator defined in (1);
λ,	third curvilinear coordinate;
μ,	variable defined in (30);
μ_1,	dynamic viscosity of fluid outside of a droplet;
μ_2,	dynamic viscosity of fluid inside of a droplet;
ξ,	first (transverse) curvilinear coordinate;
σ,	$= \sigma(c)$, non-dimensional diffusion coefficient;
$\tau, \tau_0, \tau_\alpha, \tau_\beta$,	new variable (4);
Φ,	kernel of integral operator(68);
φ,	function defining the boundary condition at the droplet surface (3);
χ,	$= \chi(\omega)$, Heaviside unit function;
Ψ,	stream function analogue; stream function;
ψ_α, ψ_β,	functions defining initial and boundary (at $\eta = 0$) conditions (3);
Ω,	function defining the principal term in the stream function expansion near the particle surface;
ω,	new (cyclic) variable (4).

1. INTRODUCTION

LINEAR problems of the unsteady state diffusion-controlled (thermal) boundary layer were the concern of a number of earlier publications (see for example [1–10]). Thus, the authors of [1], by applying the Laplace transformation with respect to time, have determined the diffusional flux to the surface of a droplet in a steady translational Stokesian flow with a sudden occurrence of chemical reaction. In [2–6], the analysis of a number of unsteady state boundary layer problems was based on the use of a self-similar variable allowing the initial convective diffusion equation to be reduced to the 1st-order partial differential equation for the diffusion boundary layer thickness and to ordinary differential equation for concentration. In [5–9], two new variables have been introduced (which simplify the initial problem) the determination of which have also required the solution of the 1st-order partial differential equation.

Introduction of new variables has made it possible to study a number of unsteady state problems described by the convective diffusion equation with time dependent coefficients [3–6, 8, 9] for the solution of which the method suggested in [1] is inapplicable. Similar problems have been analyzed in [10] by applying the Fourier sine transformation with respect to the transverse coordinate. In [11, 12], some non-linear unsteady state problems were analyzed by invoking three (four) new independent variables. Linear problems involving the 1st-order bulk chemical reaction were considered in [13].

The present paper suggests a general method for the solution of this type of problems which enables one to study the respective equations in a single form.

2. CHOICE OF COORDINATE SYSTEM. STATEMENT OF THE PROBLEM

Assuming that the fluid flow field has been found from the solution of the respective problem on hydrodynamic flow, introduce a local orthogonal system of coordinates ξ, η and λ associated with the body (droplet) surface and flow geometry in the fashion similar to that described in [14]. To do this, determine the direction of the unit vectors $\mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\lambda$, at any point M located near the particle (Fig. 1). The point M', nearest to M on the body surface, specifies the direction of $\mathbf{e}_\xi$, while the segment $|MM'|$ determines the non-dimensional coordinate $\xi = \xi(|MM'|)$, where the length scale is the characteristic body dimension (e.g. sphere radius). Without the limitation of generality, the coordinate ξ is chosen so that $\xi = 0$ corresponds to the body surface and $(g_{\xi\xi})_{\xi=0} = 1$. The direction of $\mathbf{e}_\eta$ is determined by the fluid velocity vector projection at the point M onto a plane normal to $\mathbf{e}_\xi$, while $\mathbf{e}_\lambda$ is selected so that the system of vectors $\mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\lambda$ forms an orthogonal right-handed triad (Fig. 1).

In this coordinate system, the fluid velocity vector at any point of a 3-dim. flow has in general the form $\mathbf{v} = \{v_\xi, v_\eta, 0\}$. Note that here, as distinguished from the stationary flow field, the directions of the unit vectors $\mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\lambda$ and the metric tensor components $g_{\xi\xi}, g_{\eta\eta}, g_{\lambda\lambda}$ are time-dependent (reckoning of the curvilinear coordinates η, and λ is of little importance for what follows and can be done in the same way as in [14]; $0 \le \lambda \le 2\pi$).

Let the stagnation point on the body surface, at which a special streamline approaches the surface and in the neighbourhood of which the normal fluid velocity component is directed toward (from) the surface, be referred to as the incidence (run-off) point

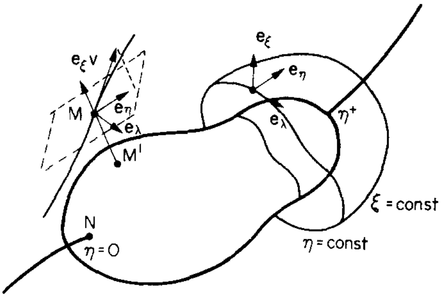

FIG. 1. Curvilinear orthogonal system of coordinates ξ, η, λ associated with the body surface and stream lines.

and the streamline originating from it, as the incidence (run-off) trajectory.

For an incompressible fluid, stream function analogue Ψ can be introduced, $\Psi(t, 0, \eta, \lambda) = 0$, so that the fluid velocity components can be expressed [14] as

$$v_\xi = -\sqrt{\frac{g_{\xi\xi}}{G}}\frac{\partial\Psi}{\partial\eta},\quad v_\eta = \sqrt{\frac{g_{\eta\eta}}{G}}\frac{\partial\Psi}{\partial\xi},$$

$$v_\lambda = 0;\quad G = g_{\xi\xi}g_{\eta\eta}g_{\lambda\lambda}.$$

In the plane and axisymmetric case, Ψ coincides with the ordinary stream function. In the case of a steady state flow, the surfaces Ψ = const. have a simple physical meaning specific of the translational flow. The incidence trajectory at a distance from the particle is a straight line then. The isolated fluid elements equally spaced from this straight line at a distance from the particle can give a clear picture of the Ψ = const. surface if followed in a fluid flow past the particle.

In the system of coordinates ξ, η, λ, the non-dimensional convective diffusion (heat conduction) equation in the boundary layer approximation is written as

$$\Lambda(c, \sigma(c)) = \frac{\partial c}{\partial t} + (\mathbf{v}\nabla)c - \varepsilon^2\frac{\partial}{\partial\xi}\sigma(c)\frac{\partial c}{\partial\xi} = 0, \quad (1)$$

$$\Psi = \xi\Omega, (\mathbf{v}\nabla)c = \frac{1}{\sqrt{g}}\left(\frac{\partial\Psi}{\partial\xi}\frac{\partial c}{\partial\eta} - \frac{\partial\Psi}{\partial\eta}\frac{\partial c}{\partial\xi}\right),$$

$$g = [G]_{\xi=0},\ \varepsilon^{-2} = Pe = a_e U_* D^{-1}$$

$$(0 < \sqrt{g}\,\Omega^{-1} < \infty, 0 < t,\quad \xi < \infty,\quad 0 < \eta < \eta\ ).\ (2)$$

Here $c_* = c_0 c$ is the concentration, c_0 characteristic concentration (e.g. concentration on the particle surface), Pe the Peclet number, a_e and U_* the characteristic dimension and velocity of particle, $D_* = D\sigma(c)$ the diffusion coefficient, $g = g(t, \eta, \lambda)$ and $\Omega = \Omega(t, \eta, \lambda)$ the familiar functions of time t and coordinates η, λ determined by the body shape and local field of velocities near its surface; $\eta = 0$ corresponds to the point in the neighbourhood of which the boundary layer is formed (for $\eta = 0$ there is the point N in Fig. 1).

The approximation (2) for the stream function analogue Ψ (linearity in ξ) is frequently used, and is valid for the boundary layer, in the problems on mass transfer of drops and bubbles (liquid films) in viscous flow of the majority of water-type liquids, and in thermal problems (potential flow), for media such as liquid metals [1–14]. The coordinate λ appears in equations (1), (2) only parametrically and is omitted in what follows.

Consider equation (1), (2) subject to the following initial and boundary conditions

$$t = 0,\quad c = \psi_\alpha(\xi, \eta);$$

$$\eta = 0,\ c = \psi_\beta(t, \xi);\ \xi = 0,\quad c = \varphi(t, \eta) \quad (3)$$

and the condition of solution boundedness at $\xi \to \infty$; here ψ_α, ψ_β, φ are some specified restricted and continuous functions.

It should be noted that in case the problems do not involve a characteristic length (see e.g. an extract solution in Section 10), it is more convenient to use a dimensional equation which can be formally derived from equation (1) at $\varepsilon = 1$ employing the replacement of σ by D_*, with corresponding dimensional quantities being taken for coordinates, time and stream function. No proviso is further made concerning this fact, and it is assumed that this operation can be performed in final results where required.

3. DESCRIPTION OF THE METHOD. EQUATIONS FOR NEW VARIABLES

In order to solve the problem (1–3), introduce (similar to [11, 12]) new variables

$$\omega = \omega(t, \eta),\quad \zeta = \varepsilon^{-1}\,\xi f(t, \eta),\quad \tau = \tau(t, \eta). \quad (4)$$

The initial equation (1), (2) will then take the form

$$\mathbf{L}\omega\frac{\partial c}{\partial\omega} + \zeta f^{-1}\left(\mathbf{L}f - \frac{1}{\sqrt{g}}\Omega'_\eta f\right)\frac{\partial c}{\partial\zeta} + \mathbf{L}\tau\frac{\partial c}{\partial\tau}$$

$$= f^2\frac{\partial}{\partial\zeta}\sigma(c)\frac{\partial c}{\partial\zeta},\quad \mathbf{L} \equiv \frac{\partial}{\partial t} + \frac{\Omega}{\sqrt{g}}\frac{\partial}{\partial\eta},\quad \Omega'_\eta \equiv \frac{\partial\Omega}{\partial\eta}. \quad (5)$$

Self-similar solutions of equation (1) can be obtained from (5) by assuming $c = c(\zeta)$ and requiring the equality (which is an equation for determining the variable ζ):

$$f^{-1}(\mathbf{L}f - g^{-1/2}\Omega'_\eta f) = f^2. \quad (6)$$

The distribution of concentration will then obey the ordinary differential equation

$$\zeta c'_\zeta = (\sigma(c)c'_\zeta)'_\zeta. \quad (7)$$

It should be noted that in [2–4] the analysis of a number of linear problems (σ = 1) was actually performed by the method based on equations (6) and (7). In [7–9], a two-parametric solution of the type $c = c(\zeta, \tau)$ was sought for linear problems.

Of course, the self-similar solutions of the type (6), (7) or the results obtained in [7–9] do not exhaust all possible solutions of equation (1).

In the general case, to analyze equation (1), (2), we shall use all three new variables (4) and impose the requirement that the functions ω, f and τ should generate the following system of partial differential equations [11, 12]:

$$\mathbf{L}\omega = 0, \quad \mathbf{L}f = g^{-1/2}\Omega'_\eta f, \quad \mathbf{L}\tau = f^2 \tag{8}$$

the general solution of which is

$$\omega = \omega(t, \eta), \quad f = Z_f(\omega)\exp\left\{\int^\eta \left(\frac{\partial}{\partial\eta}\ln\Omega\right)_{(\omega)} \mathrm{d}\eta\right\},$$

$$\tau = \int^\eta (\sqrt{g}\Omega^{-1}f^2)_{(\omega)}\,\mathrm{d}\eta + Z_\tau(\omega).$$

Here and subsequently the notation $S_{(\omega)}\,\mathrm{d}\eta$ (or $S_{(\omega)}\,\mathrm{d}t$) in the integrand means that the function $S(t, \eta)$ is written by the expression for $\omega = \omega(t, \eta)$ in terms of the variables ω and η (or ω and t) and, in performing the integration, ω is taken to be a parameter; ω is any non-trivial solution of equation (8), Z_f and Z_τ are arbitrary functions depending upon ω alone.

After employing substitution (4) and (8), equations (1) and (2) take the form:

$$\mathbf{A}(\tau, \zeta)c \equiv \frac{\partial c}{\partial \tau} - \frac{\partial}{\partial \zeta}\sigma(c)\frac{\partial c}{\partial \zeta} = 0, \quad c = c(\omega, \zeta, \tau). \tag{9}$$

Equation (9) is much simpler than the initial one, (1), (2), and does not involve the variable ω which is quite analogous to the cyclic variables in analytical mechanics. The appropriate initial and boundary conditions (3) should be written in terms of the variables (4) and (8), which reduces the initial boundary value problem (1–3) to an ordinary heat conduction equation depending on the parameter ω.

The transformation (4), (8), $(t, \xi, \eta) \to (\omega, \zeta, \tau)$ (if it is non-degenerate) is complete in the sense that no loss of any solutions to the initial equations (1) and (2) occurs when passing to equation (9).

4. CHOICE OF VARIABLES. INITIAL AND BOUNDARY CONDITIONS

In choosing the specific form of the variables (4) and (8), it is necessary to distinguish between the two cases:

$$\text{(a)}\ \lim_{\eta \to 0} (\eta\sqrt{g}\Omega^{-1}) = 0,$$

$$\text{(b)}\ \lim_{\eta \to 0} (\eta\sqrt{g}\Omega^{-1}) > 0,$$

in which the integral curves of characteristic equation

$$t'_\eta = \sqrt{g}\Omega^{-1}, \tag{10}$$

corresponding to the operator $\mathbf{L}$, show a substantially dissimilar behaviour. In the first case it is a non-trivial integral curve of equation (10) which passes through the coordinate origin in the plane η, t (Fig. 2), while in the second case such an integral curve does not exist (Fig. 3).

The Case (b) usually holds for the problems when $\eta = 0$ corresponds to the stagnation point on the body surface (point N in Fig. 1).

Case (a). The first two variables (4), (8) throughout the flow region are selected in the form

$$\omega(t, \eta) = -t + t(\eta),$$

$$f = \exp\left\{\int_{\eta_*}^{\eta} \left(\frac{\partial}{\partial\eta}\ln\Omega\right)_{(\omega)} \mathrm{d}\eta\right\}. \tag{11}$$

Here $t(\eta)$ is the solution of the characteristic equation (10) restricted by the property $t(0) = 0$; $\eta_* \in [0, \eta^+)$ is anything.

In choosing the variable τ, (4) and (8) it should be noted that in the region $\omega < 0$ the integral curves of the characteristic equation (10) intersect only the axis t and therefore the solution in this region depends only upon the boundary condition (3) at $\eta = 0$ and does not depend upon the initial condition at $t = 0$; in the region $\omega > 0$ the integral curves intersect only the axis η and here the solution depends only upon the initial condition (3) at $t = 0$ and does not depend upon the boundary condition at $\eta = 0$ (Fig. 2). This means that distribution of concentration on one side of the curve $\omega = 0$ is not affected by the initial, and on the other by the boundary, condition.

It should be noted that equations (1), (2) have the particular solution $c = Z(\omega)$ (Z is an arbitrary function), which, at $\psi_\alpha = \psi_\gamma(\eta)$, $\psi_\beta = \psi_\beta(t)$ and inert boundary conditions $\partial c/\partial\xi = 0$ on the particle surface $\xi = 0$, may conform to the initial and boundary (at $\eta = 0$) conditions (3) due to the choice of the function Z (Z can be a discontinuous one; thus, at $t = 0, c = 1$; $\eta = 0, c = -1$ we have $c = \operatorname{sign}\omega$). Because of this situations may occur in the boundary layer problems

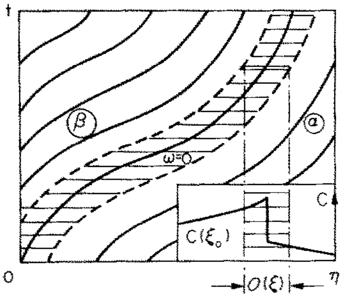

FIG. 2. Behaviour of integral curves of the characteristic equation corresponding to the operator L (Case a).

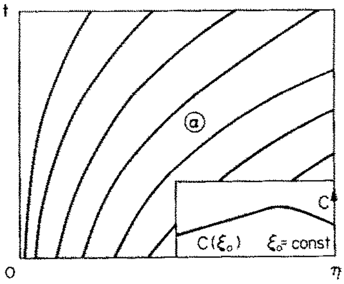

FIG. 3. Behaviour of integral curves of the characteristic equation (10) in Case (b).

of unsteady state convective diffusion which involve concentration discontinuity as a consequence of the hyperbolic nature of the initial equation (1) with respect to the variables t and η.

With the above in mind, let us introduce two time dependent similar variables τ_α and τ_β (here and subsequently the quantities associated with the region $\omega > 0$ are indicated by the subscript α, and with the region $\omega < 0$, by the subscript β), either of which is continuous in its own domain, satisfies the last of equations (8) and is determined by the boundary conditions

$$t = 0, \quad \tau_\alpha = 0; \quad \eta = 0, \quad \tau_\beta = 0. \tag{12}$$

The solution of problems (8) and (12) is:

$$\tau_\alpha = \tau_\alpha(t, \eta) = \int_0^t f^2{}_{(\omega)} \mathrm{d}t = \tau_\beta - \tau_\beta(0, T(\omega)), \tag{13}$$

$$\tau_\beta = \tau_\beta(t, \eta) = \int_0^\eta (\sqrt{g}\Omega^{-1}f^2)_{(\omega)} \mathrm{d}\eta; \; T(\omega(0, \eta)) \equiv \eta. \tag{14}$$

Expressing the former variables t, ξ, η in terms of the new ones ω, ζ, τ (4), (8), (13) and (14) in each region of one-signed ω (the Jacobian of this transformation differs from zero) and substituting them into (1–3) for the concentration distribution yield:

$$c = \chi(\omega)c_\alpha(\omega, \zeta, \tau_\alpha) + \chi(-\omega)c_\beta(\omega, \zeta, \tau_\beta) \tag{15}$$

$$\chi(\omega) = \begin{cases} 0, \text{ at } \omega < 0, \\ 1, \text{ at } \omega > 0. \end{cases}$$

Here $\chi(\omega)$ is the Heaviside unit function, while the functions c_α and c_β are determined by the solutions of the following boundary value problems:

$$\mathbf{A}(\tau_\gamma, \zeta)c_\gamma = 0; \quad \tau_\gamma = 0, \quad c_\gamma = \psi_\gamma(\omega, \zeta); \tag{16}$$

$$\zeta = 0, c_\gamma = \varphi_\gamma(\tau_\gamma, \omega); \gamma = \alpha, \beta; \varphi_\gamma(\tau_\gamma, \omega) \equiv$$

$$\varphi(t(\tau_\gamma, \omega), \eta(\tau_\gamma, \omega)); \psi_\alpha \equiv \psi_\alpha(\varepsilon\zeta f^{-1}(0, T(\omega)),$$

$$T(\omega)); \psi_\beta \equiv \psi_\beta(-\omega, \varepsilon\zeta f^{-1}(-\omega, 0)).$$

It is seen that the initial problem (1–3) with the variable coefficients has simplified substantially to the equation with constant coefficients (9) which depends only upon two variables, ζ and τ (and no longer on ω), and to the boundary conditions (16), which depend parametrically upon the third cyclic coordinate ω.

Equation (9) occurs frequently in the theory of non-linear heat conduction [15, 16] and filtration [17] and there are already its exact analytical, approximate or numerical solutions (under certain initial and boundary conditions) for many functions $\sigma = \sigma(c)$ (see Section 6).

In general, the solution of the problem (1–3) becomes discontinuous in passing through the characteristic $\omega(t, \eta) = 0$ (11) (Fig. 2). For this reason, the diffusional boundary layer approximation (1–3) by itself proves to be unjustified close to the front $\eta = T(t)$ moving with the velocity $u = (g^{-1/2}\,\Omega)_{t, \eta = T(t)}$ within the region $|\omega| \leq O(\varepsilon)$, $\varepsilon \ll 1$. Physically this implies the presence of large concentration gradients in the region and necessitates the use of complete convective diffusion equation in the vicinity of $\omega = 0$. The problem on concentration distribution in the neighbourhood of the front $\eta = T(t)$ can be formulated by the method of matched asymptotic series expansions [12, 18].

It should be noted that even though the boundary layer solution is inapplicable in the region $|\eta - T(t)| \leq O(\varepsilon)$, it provides, when used at $\eta^+ = O(1)$, a correct result for the principal term of asymptotic expansion (in the parameter $\varepsilon \ll 1$) for the basic characteristic of the process of transport, i.e. the integral inflow of the reacting substance (heat) to the body surface.

Case (*b*). Here all of the integral curves of the characteristic equation (10) intersect the axis t alone and keep clear of the axis η (Fig. 3). This implies that only the condition (3) at $t = 0$ can here be satisfied, while the boundary condition at $\eta = 0$ for the arbitrary function ψ_β can no longer be imposed [just as the second condition for the variable τ_β in (12)]; here, one can only require compliance with the (weak) condition of boundedness for the solution $|c| < \infty$ at $\eta \to 0$ (or in an axisymmetric case—the condition of axial symmetry $\partial c/\partial \eta = 0$ at $\eta = 0$). In Case (b), the diffusional boundary layer equations (1) and (2) are uniformly applicable in a small parameter ε (and its solutions are smooth) to the entire flow region. The variables ω, ζ, τ (4) and (8) should also be chosen in the form of (11) and (13), with the difference that the condition $t(0) = 0$ does not hold now [there being instead the condition $t(\eta_*) = 0$, with $\eta_* \in (0, \eta^+)$ anything]. With the replacement such as this, the problem (1–3), subject to the condition of solution boundedness instead of the boundary condition at $\eta = O(3)$, is reduced to (15) at $\gamma = \alpha$ and all what has just been said for Case (a) will also hold here. The solution of the respective non-linear problem (1–3) in this case is prescribed by the first term of equations (15) and (16):

$$c = c_\alpha(\omega, \zeta, \tau_\alpha). \tag{17}$$

In Case (b) it is also meaningful to consider the solutions to equations (1) and (2) that satisfy the boundary condition at infinity at $\xi \to \infty$ instead of the initial condition at $t = O(3)$ [8, 9] (see the comment at the end of section 6).

It should be noted that to choose the variables (4) instead of (11), (13) and (14) for the solution of particular problems it is frequently more convenient to directly employ equation (8).

5. LINEAR CASE. PROBLEMS WITH VOLUMETRIC REACTION

In the linear case ($\sigma = 1$) the problem (1–3) is reduced to a respective boundary value problem for an ordinary linear heat conduction equation with constant coefficients (9), (16) the solution of which is well known. The concentration (temperature) distribution is then prescribed by formula (15) [or (17)] where the functions c_α and c_β are determined by:

$$c_\gamma = \frac{1}{2\sqrt{\pi\tau_\gamma}}\int_0^\infty \left\{\exp\left[-\frac{(\zeta-\zeta^*)^2}{4\tau_\gamma}\right]-\exp\left[-\frac{(\zeta+\zeta^*)^2}{4\tau_\gamma}\right]\right\} \times \psi_\gamma(\omega,\zeta^*)\,\mathrm{d}\zeta^* + \frac{1}{2\sqrt{\pi}}\int_0^{\tau_\gamma}\frac{\zeta}{(\tau_\gamma-\lambda)^{3/2}} \times \exp\left[-\frac{\zeta^2}{4(\tau_\gamma-\lambda)}\right]\times\varphi_\gamma(\omega,\lambda)\,\mathrm{d}\lambda;\ \gamma=\alpha,\beta. \tag{18}$$

Equations (15) and (18) extend the results of [1–10] to the arbitrary initial and boundary conditions (3).

The problems involving the 1st-order bulk reaction are described by

$$\Lambda(c,1) = -kc, \quad k = k'a_e U^{-1} \tag{19}$$

by equation (2) and initial and boundary conditions (3). Here k' is the reaction rate constant.

In this case the substitution

$$c = \mathrm{e}^{-kt}u \tag{20}$$

reduces the initial problem (2), (3) and (19) to the following boundary value problem to determine the function u

$$\Lambda(u,1)=0;\ t=0,\quad u=\psi_\alpha^*;\ \eta=0,\quad u=\psi_\beta^*;$$
$$\xi=0,\quad u=\varphi^*,\quad \psi_\alpha^*=\psi_\alpha,\quad \psi_\beta^*=\mathrm{e}^{kt}\psi_\beta,\quad \varphi^*=\mathrm{e}^{kt}\varphi. \tag{21}$$

It is seen from comparison of equations (1) and (21) that in the case of the 1st-order bulk reaction the problem (2), (3) and (19) has reduced to the solution of an auxiliary problem for the function u with the volumetric reaction absent. The function u is therefore described by (15) and (18) with the respective replacement of c by u and ψ_α, ψ_β, φ by ψ_α^*, ψ_β^*, φ^*, whereupon concentration distribution is determined by (20).

Note, that if the right-hand side of equation (1) contained the source $q = q(t,\eta,\xi)$, then the right-hand side of the final equation (9), upon transformations (4) and (8), would have the term qf^{-2}. In the linear case ($\sigma = 1$) the solution of such a non-uniform problem for equation (9) is well known and therefore is dropped from consideration.

6. EXACT SOLUTIONS OF SOME NONLINEAR BOUNDARY VALUE PROBLEMS

A wide class of non-trivial solutions for the non-linear boundary value problem (1–3) can be obtained for the following functions occurring in the initial and boundary conditions (3):

$$\psi_\alpha = \psi_\alpha(\eta),\quad \psi_\beta = \psi_\beta(t);\ \mathrm{L}\varphi = 0. \tag{22}$$

The last condition in (22) implies that the boundary condition at $\xi = 0(3)$, written in terms of the variables (4), (8), depends only upon the cyclic variable ω, i.e. $\varphi = \varphi(\omega)$; in particular, this condition is satisfied by $\varphi = \mathrm{const}$.

By virtue of (9), (15) and (16), the solution of the problem (1–3) and (22) as well as the local diffusional flux to a reacting surface in Case (a) can be presented as

$$c = \chi(\omega)c_\alpha + \chi(-\omega)c_\beta, \tag{23}$$

$$c_\alpha = Y\left(\frac{\zeta}{2\sqrt{\tau_\alpha}},\ \varphi(\omega),\ \psi_\alpha[T(\omega)]\right),$$

$$c_\beta = Y\left(\frac{\zeta}{2\sqrt{\tau_\beta}},\ \varphi(\omega),\ \psi_\beta(-\omega)\right),$$

$$j = \left(\frac{\partial c}{\partial\xi}\right)_{\xi=0} = \frac{\chi(\omega)f}{2\varepsilon\sqrt{\tau_\alpha}}J(\varphi(\omega),\ \psi_\alpha[T(\omega)]) + \frac{\chi(-\omega)f}{2\varepsilon\sqrt{\tau_\beta}}J(\varphi(\omega),\ \psi_\beta(-\omega)),$$

$$J(a,b) \equiv J(\sigma;a,b) = [\sigma(Y)Y'_x]_{x=0}. \tag{24}$$

Here, the function $Y(x,a,b)$ is the solution of the following ordinary differential equation

$$[\sigma(Y)Y'_x]'_x + 2xY'_x = 0;\ Y(0) = a;\ Y(+\infty) = b, \tag{25}$$

which depends upon the parameters a and b.

In Case (b) the solution of the problem (1–3) and (22) is prescribed by $c = c_\alpha$, where c_α is defined in (23).

Equation (25) occurs frequently in the problems of non-linear heat conduction and filtration [15–17] and its solution has already been obtained for many relations $\sigma = \sigma(c)$.

It should be noted that since the form of the variables (4) and (8) does not depend upon the diffusion coefficient (thermal diffusivity) $\sigma(c)$, solutions of a number of specific problems can be obtained by directly combining familiar solutions of equation (25) for different modes of flow past droplets, bubbles, liquid films, etc. (see e.g. [1–12]). The mode of flow will then determine only the form of the variables (4), (8) entering into equations (23), (24).

With the above discussion in mind, let us point out the solutions of the problem (25) which appear for some frequently encountered specific relations $\sigma = \sigma(c)$; Sections 8–10 are devoted to some physical statements of the problems which correspond to different types of flow and define the form of the variables (4) and (8).

In the simplest linear case, at the constant diffusion coefficient $\sigma = \sigma_0$, the solution of equation (25) is

$$Y(x,a,b) = a + (b-a)\,\mathrm{erf}\left(\frac{x}{\sqrt{\sigma_0}}\right). \tag{26}$$

Now consider the most frequently encountered linear dependence (approximation) of the diffusion coefficient (thermal diffusivity) on concentration

$$\sigma(c) = \sigma_0(1+\kappa c). \tag{27}$$

The replacement

$$z = \frac{x}{\sqrt{\sigma_0(1+\kappa b)}},\quad u = \frac{1+\kappa Y}{1+\kappa b} \tag{28}$$

reduces the problem (25) and (27) to

$$(uu'_z)'_z + 2zu'_z = 0;\ \S u(0) = \lambda, \quad u(\infty) = 1,$$
$$\lambda = (1 + \kappa a)(1 + \kappa b)^{-1}. \quad (29)$$

Introduction of the new variable [17] makes it possible to represent the solution of (29) in parametric form as

$$u = u(\mu) = \frac{\mathrm{d}z}{\mathrm{d}\mu}, \quad z = z(\mu) = \int_0^\mu u\,\mathrm{d}\mu, \qquad (30)$$

where the function $z(\mu)$ is the solution of an auxiliary problem

$$\frac{\mathrm{d}^3 z}{\mathrm{d}\mu^3} + 2z\frac{\mathrm{d}^2 z}{\mathrm{d}\mu^2} = 0;\ z(0) = 0,$$
$$z'_\mu(0) = \lambda, \quad z'_\mu(\infty) = 1. \quad (31)$$

The problem (31) is frequently encountered in the theory of hydrodynamic boundary layer; for $\lambda = 0$ its solution was obtained by Blasius (see e.g. [19]). The case $\lambda \neq 0$ was considered in [20, 21].

For practical purposes the greatest interest attaches to an equation for the diffusional flux (24) which is defined by the expression for J. Using the formula $(\mathrm{d}u/\mathrm{d}z)_{z=0} = (u'_\mu/z'_\mu)_{\mu=0}$ and the results of [20] we obtain

$$J(a, b) = \sqrt{\sigma_0}\,\kappa^{-1}(1 + \kappa b)^{3/2} R\left(\frac{1 + \kappa a}{1 + \kappa b}\right). \quad (32)$$

Here the function $R = R(\lambda)$ is defined by the equality $z''_{\mu\mu}(0) = R(\lambda)$, has the properties $R(0) = 0.33206$, $R(1) = 0$ and is shown in Fig. 4.

At $\lambda \approx 1$ the function R is determined from the asymptotic equation [20]

$$\lambda^{-1} - 1 = 1.772\xi - 0.59\xi^2, \quad \xi = \lambda^{-3/2}R.$$

Note that the problem (29) has been studied by numerical and approximate analytical methods in [17, 22].

In a number of problems (associated, for instance, with adsorption) the following expression for the diffusion coefficient can be met

$$\sigma(c) = \sigma_0(1 - \kappa c)^{-1}\ (\kappa \geq 0). \qquad (33)$$

According to [16, 23] the solution of the problem (25) and (33) can be presented as

$$Y = \kappa^{-1}[1 - (1 - \kappa b)\mathrm{e}^{-2E}],$$
$$x = \mathrm{e}^E\left[\frac{\sigma_0}{2P(1 - \kappa b)}\right]^{1/2}\left[w - \left(\frac{\partial E}{\partial w}\right)^{-1}\right]$$
$$E = E(w, P) = \int_0^w (\lambda^2 - P\ln\lambda^2)^{-1/2}\,\mathrm{d}\lambda,$$
$$P = P\left(\frac{1 - \kappa a}{1 - \kappa b}\right), \quad (1 \leq w \leq +\infty). \qquad (34)$$

The dependence of $Y = Y(w)$, $x = x(w)$ on w(34) parametrically determines the form of the function $Y = Y(x)$, with the value of the parameter $P = P(Q)$ being found by solving the transcendental equation

$$\ln Q = -2E(1, P) \quad [P(0) = 0]. \qquad (35)$$

Figure 5 shows the function $P = P(Q)$ which corresponds to equation (35).

Taking into account that to the reacting surface $x = 0$ there corresponds the value of the parameter $w = 1$, we obtain the following equation for the quantity J, which defines the diffusional flux (24),

$$J(a,b) = \kappa^{-1}[8\sigma_0(1 - \kappa a)]^{1/2}\left[P\left(\frac{1 - \kappa a}{1 - \kappa b}\right)\right]^{-1/2}. \qquad (36)$$

An exact analytical solution of the problem (25) at $b = 0$ is suggested in [23] for the following dependence of the diffusion coefficient upon concentration (see also [16]):

$$\sigma(c) = \sigma_0(1 + \kappa c + vc^2)^{-1}.$$

For thermal problems, in the case of exponential dependence of thermal diffusivity upon temperature,

$$\sigma(c) = \sigma_0 c^n, \quad n > 0 \qquad (37)$$

the solution of the problem (25) and (37) at $b = 0$ is of the form (see e.g. [11, 24]):

$$Y(z) = \begin{cases} a(1 - z)^{1/n}\,p(z, n)\,p^{-1}(0, n) & \text{at } 0 \leq z \leq 1 \\ 0 & \text{at } 1 \leq z \leq \infty \end{cases} \qquad (38)$$

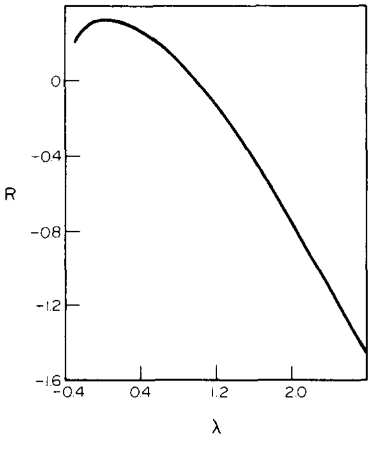

FIG. 4. The curve $R = R(\lambda)$ [20].

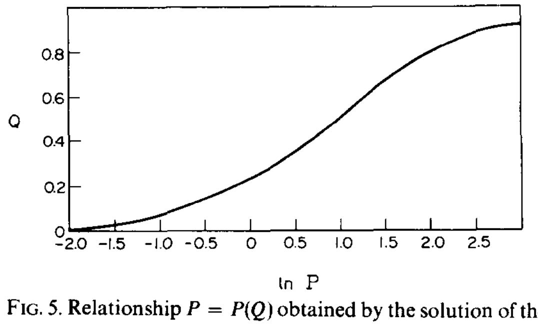

FIG. 5. Relationship $P = P(Q)$ obtained by the solution of the transcendental equation (35).

$$z = \frac{x}{q(n)\sqrt{2\sigma_0 a^n}}, \quad p(z,n) = \sum_{k=0}^{\infty} \alpha_k z^k,$$

$$q(n) = [np^n(0,n)]^{-1/2}, \quad \alpha_0 = 1,$$

$$\alpha_1 = -\frac{1}{2n(n+1)},$$

$$\alpha_2 = -\alpha_1 \frac{1 + 0.5\alpha_1(6n^2 + n - 3)}{3(2n+1)}, \quad \dots .$$

The function (38) is a generalized solution of the problem (25), (36) and has a discontinuous derivative at $z = 1$ (the flux $\sigma(c)\partial c/\partial\xi$ in this case remaining continuous); $q(1) = 1.143$ [17].

A detailed description of the procedure of numerical integration of equation (25) as well as the results of integration for the case of exponential dependence of the diffusion coefficient upon concentration, $\sigma(c) = \sigma_0 \exp(\kappa c)$, are presented in [25].

When the diffusion coefficient depends arbitrarily upon concentration, one can use numerical or approximate analytical methods for the solution of the problem (25) [15, 16]. Thus, one can employ the following approximate iterative method to determine the function Y [15, 16]

$$Y_{\mathrm{m}}(x,a,b) = a + (b-a)H\left(\frac{x}{\sqrt{\sigma(b)}}\right)H^{-1}(+\infty);$$

$$m = 0, 1, \dots \qquad (39)$$

$$H(x) = \int_0^x \exp\left[-\int_0^x \frac{2x\,\mathrm{d}x}{\sigma_*}\right]\frac{\mathrm{d}x}{\sigma_*}, \quad \sigma_* = \frac{\sigma(b)}{\sigma(Y_{m-1})}$$

Here, as an initial iterate Y_0 we take the solution (26) which corresponds to the constant diffusion coefficient. At $m = 1$ equation (39) yields rather satisfactory results at small values of the derivative $|\partial\sigma/\partial c| \ll 1$.

Moreover, when the diffusion coefficient is restricted over the entire concentration range, the function $\sigma = \sigma(c)$, depending upon its form, can be approximated according to equation (27) or (33) by a suitable choice of the free parameters σ_0 and κ.

Remark. For an unsteady state flow in Case (b) it is a sound plan to consider also the solutions of equations (1) and (2) satisfying constant boundary conditions on the particle surface and at infinity. Then the equation

$$c_\beta = Y\left(\frac{\zeta}{2\sqrt{\tau_\beta}}, a, b\right)$$

yields the solution of the problem (1) and (2) subject to the boundary conditions

$$\xi = 0, \quad c = a; \xi \to \infty, \quad c \to b(a, b = \text{const.})$$

and the initial condition which corresponds to stationary concentration distribution in the flow $\Omega(0, \eta)$ at $t \le 0$. This type of a problem describes mass and heat transfer of the particle moving with a variable velocity at $t > 0$ which was initially at rest [8, 9].

7. COMPUTATION OF THE MEAN SHERWOOD NUMBER AND OF THE INTEGRAL DIFFUSIONAL FLUX TO A REACTING SURFACE

The mean Sherwood number Sh and dimensionless integral diffusional flux I are determined by

$$I = Sh\,S = \int_s j\,\mathrm{d}s \left(S = \int_s 1\,\mathrm{d}s\right). \qquad (40)$$

Here s and S are the surface and (dimensionless) area of the reacting particle surface.

Let us consider the specific case of the initial and boundary conditions (3), (22)

$$\psi_\alpha = A_\alpha, \quad \psi_\beta = A_\beta, \quad \varphi = B(A_\alpha, A_\beta, B = \text{const.}). \qquad (41)$$

Now, let us keep in mind that the local diffusional flux j^0, corresponding to the linear problem $\sigma = 1$, (24) and (26), in Case (a) at $A_\alpha = A_\beta = 1, B = 0$ in (22), (41) has the form

$$j^0 = \chi(\omega)j_\alpha^0 + \chi(-\omega)j_\beta^0,$$

$$j_\gamma^0 = \varepsilon^{-1}f(\pi\tau_\gamma)^{-1/2}; \gamma = \alpha, \beta. \qquad (42)$$

Comparison of equations (24), (41) and (42) gives the following expression for the local diffusional flux to the particle at an arbitrary dependence of the diffusion coefficient on concentration for any values of the parameters A_α, A_β and B in (41)

$$j = \frac{\sqrt{\pi}}{2}\chi(\omega)J(B, A_\alpha)j_\alpha^0 + \frac{\sqrt{\pi}}{2}\chi(-\omega)J(B, A_\beta)j_\beta^0.$$

Integration of this relation over the particle surface s yields

$$I = Sh\,S = \chi(\omega)I_\alpha + \chi(-\omega)I_\beta,$$

$$I = I(\sigma, A_\alpha, A_\beta, B, t),$$

$$I_\alpha = \frac{\sqrt{\pi}}{2}J(B, A_\alpha)I_\alpha^0, \qquad (43)$$

$$I_\beta = \frac{\sqrt{\pi}}{2}J(B, A_\beta)I_\beta^0.$$

Here $I^0 = \chi(\omega)I_\alpha^0 + \chi(-\omega)I_\beta^0$ is the integral diffusional flux corresponding to the linear problem $\sigma = 1$ at $A_\alpha = A_\beta = 1$, $B = 0$, i.e. $I^0 = I(1, 1, 1, 0, t)$.

In Case (b), the formula $I = I_\alpha$, where I_α is defined in (43), should be used for the integral flux.

Equations (43) allow an effective calculation of the integral diffusional flux and mean Sherwood number when the diffusion coefficient is an arbitrary function of the concentration provided appropriate expressions for the linear problem ($\sigma = 1$) are known as well as the functions J (24) which are determined by solving the ordinary differential equation (25). Thus, for the specific cases of the diffusion coefficient dependence upon concentrations (27) and (33) the functions J are determined by equations (32) and (36). Moreover, a number of linear problems are available at present for which the value of I^0 has already been calculated (see e.g. [1, 2, 7–9]).

8. UNSTEADY STATE DIFFUSION TO A SPHERE (DROPLET) IN TRANSLATIONAL AND SHEAR FLOW

Consider now some specific cases when the function Ω (2) can be represented as a product of two factors.

$$\Omega(t,\eta) = U(t)X(\eta), \quad g = g(\eta). \tag{44}$$

In this case it is convenient that the variables ω and f, (4) and (8) be taken in the form

$$\omega = -\int_0^t U\,\mathrm{d}t + \int_{\eta_*}^{\eta} \sqrt{g}X^{-1}\,\mathrm{d}\eta, \quad f = X(\eta). \tag{45}$$

Formulae (45) are more practical than (11); they can be obtained directly from equation (8) (see remark at the end of section 4).

It should be noted that if any specific solution of the last equation from (8) is known τ_0 (e.g. τ_α or τ_β), the variables τ_α and τ_β (13), (14) for the solution of specific problems can be also determined in another manner on the following simple grounds. For any function Z the expression $\tau_0 + Z(\omega)$ is also the solution for the last equation from (8). Therefore, by virtue of the boundary conditions (12) the variables τ_α and τ_β are determined by $\tau_{\alpha,\beta} = \tau_0 + Z_{\alpha,\beta}(\omega)$, where the explicit form of the functions Z_α and Z_β is found by solving the equations

$$[\tau_0 + Z_\alpha]_{t=0} = 0 \text{ and } [\tau_0 + Z_\beta]_{\eta=0} = 0.$$

As an illustration of the method suggested let us determine the quantities occurring in (22) and (23) for the unsteady state heat and mass transfer of a sphere moving in an ideal liquid with the velocity $U_0U(t)$

$$U(t) = (1+\gamma t)^{-1}, \quad \gamma \geq 0. \tag{46}$$

The dimensionless variables and the coefficients in equation (1) and (2) are determined then [7] by

$$\xi = r - 1, \quad g = \sin^2\theta, \quad \Omega = \frac{1}{2}U(t)\sin^2\theta, \tag{47}$$

where the characteristic scales are taken to be the sphere radius a_e and its tripled velocity at time zero, $U_* = 3U_0$; r, θ is the spherical coordinate system fixed at the sphere center; the angle θ is calculated from the forward stagnation point of the sphere.

Suppose one side of the sphere surface $0 \leq \theta \leq \theta_0$, $0 \leq \theta_0 \leq \pi$ in the vicinity of the flux incidence point contains a varnished inclusion (Fig. 6) (which corresponds to the boundary condition $\xi = 0, 0 \leq \theta \leq \theta_0$, $\partial c/\partial\xi = 0$), while the other side, in the region $\theta_0 \leq \theta \leq \pi$, absorbs the substance completely which corresponds to $\varphi = 0$ in the boundary conditions (3) and (22).

Using equations (13), (14), (44) and (45) for the variables (4), we obtain ($\eta_* = \theta_0$):

$$\omega(t,\theta,\theta_0) = \omega_*(t,\theta) - \omega_*(0,\theta_0), \quad f = \frac{1}{2}\sin^2\theta,$$

$$\omega_*(t,\theta) = \frac{1}{\gamma}\ln U(t) + \ln \mathrm{tg}^2\frac{\theta}{2}(\eta = \theta - \theta_0),$$

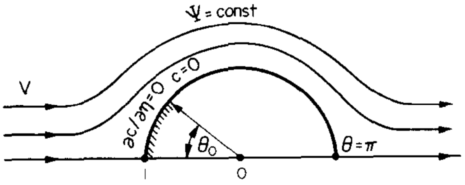

FIG. 6. Scheme of flow around a sphere (hatched area corresponding to a varnished inclusion).

$$\tau_\alpha = 4\mathrm{e}^{-\gamma\omega_*}\left[B(S(\omega_*), 2+\gamma, 2-\gamma) - B\left(\cos^2\frac{\theta}{2}, 2+\gamma, 2-\gamma\right)\right], \tag{48}$$

$$\tau_\beta = 4\mathrm{e}^{-\gamma\omega_*}\left[B\left(\cos^2\frac{\theta_0}{2}, 2+\gamma, 2-\gamma\right) - B\left(\cos^2\frac{\theta}{2}, 2+\gamma, 2-\gamma\right)\right],$$

$$S(\omega_*) = (1+\mathrm{e}^{\omega_*})^{-1},$$

$$B(x,p,q) = \int_0^x \lambda^{p-1}(1-\lambda)^{q-1}\,\mathrm{d}\lambda.$$

Here $B(x, p, q)$ is the incomplete beta-function.

In the specific case of a steady state flow $U(t) \equiv 1$, taking into account

$$\lim_{\gamma\to 0}[U(t)]^{-1/\gamma} = \mathrm{e}^t$$

and taking the limit as $\gamma \to 0$ in (46), we obtain from (48)

$$\tau_\alpha = \tau(\cos\theta) - \tau\left(-\mathrm{th}\frac{\omega_*}{2}\right),$$

$$\tau_\beta = \tau(\cos\theta) - \tau(\cos\theta_0)$$

$$\tau(x) = \frac{1}{6}(2+x)(1-x)^2,$$

$$\omega_* = -t + \ln \mathrm{tg}^2\frac{\theta}{2}. \tag{49}$$

Equations (47) and (49) at $U \equiv 1$ also correspond to the unsteady state diffusion to a spherical droplet in a translational Stokesian flow (the Hadamard–Rybchinsky stream function) moving with the velocity U_0 at infinity provided the characteristic scale is taken to be $U_* = U_0\,\mu_1(\mu_1 + \mu_2)^{-1}$, where μ_1 and μ_2 are dynamic liquid velocities inside and outside of the droplet.

In the simplest linear case $\sigma = 1$, when motion of a sphere in liquid is governed by (46) and (47) for the case (41) at $B = 0$, the solution of the problem (1–3) and (47) in the region $\theta_0 \leq \theta \leq \pi$ is, by virtue of (48), of the form

$$c = A_\alpha\chi(\omega)\,\mathrm{erf}\left(\frac{(r-1)\sin^2\theta}{4\varepsilon\sqrt{\tau_\alpha}}\right)$$

$$+ A_\beta \chi(-\omega) \operatorname{erf}\left(\frac{(r-1)\sin^2\theta}{4\varepsilon\sqrt{\tau_\beta}}\right) \quad (50)$$

$$(0 \le \theta \le \theta_0, \quad c = A_\beta).$$

Now, investigate the limiting behaviour of the solution (50) at $\theta_0 \to 0$. Taking into consideration the property

$$\lim_{\theta_0 \to 0} \omega(t, \theta, \theta_0) = +\infty$$

and proceeding to the respective limit in (50), we obtain that concentration distribution in this case is continuous and is described only by the first term of equation (50) [which corresponds to $A_\beta = 0$ in (50)]. This means that at $\theta_0 = 0$ (which is consistent with the absence of a varnished inclusion on the sphere surface), the concentration distribution in the flow is fully determined by the initial condition (3) alone, i.e. one cannot impose at a time the initial condition at $t = 0$ and the boundary condition at $\theta = 0$.

The above example is an apt illustration of the results of Section 4; at $\theta_0 \neq 0$ the problem (1–3), (46) and (47) corresponds to Case (a), while at $\theta_0 = 0$, to Case (b).

In a similar fashion one can show that the function

$$c = h\left(-\operatorname{th}\frac{\omega_*}{2}\right)\operatorname{erf}\left(\frac{(r-1)\sin^2\theta}{4\varepsilon\sqrt{\tau_\alpha}}\right)$$

[where τ_α is determined in (49)] is the solution to (1), (2) and (47) at $U(t) \equiv 1$ corresponding to a non-uniform initial distribution of concentration in the fluid flow $t = 0$, $c = h(\cos\theta)$ and to the condition of complete substance absorption on the sphere surface. This expression generalizes the results of [1, 2, 6] where the case $h = \text{const.}$ was considered.

When a sphere moves with a variable velocity $U(t)$, the variables (4) are determined from equations (44), (45) and (13), (14). The respective solutions of non-linear problems (1–3), (27), (33), (37), (46) and (47) are specified by equations (23), (30), (31), (34), (36), (38) and (48).

It should be noted that equations (46) and (48) can be used as a basis for approximate integration of equations (1–3) and (47) in case of an arbitrary function of sphere motion, $U(t)$. For this purpose, by differentiating (46) with respect to t, determine the parameter γ in terms of the function $U(t)$

$$\gamma = \mathrm{d}U^{-1}(t)/\mathrm{d}t. \quad (51)$$

It is the substitution of the expression $\gamma = \gamma(U(t))$ into (48) that yields approximate relationships for the variables (4). In the particular case of the steady state motion of a sphere or motion following (46), equations (48) and (51) are exact.

The integral diffusional flux I^0 (at $\theta_0 = 0$), corresponding to the stationary flow field $U \equiv 1$ at $\sigma = 1$, in the case of sudden reaction occurrence is calculated in [1, 2, 7]. This allows determination of the integral flux I from equations (24), (32), (36) and (43) in non-linear problems corresponding to (27) and (33).

For a droplet in a steady state shear Stokesian flow (Case b), the functions occurring in (1) and (2) have the form

$$\xi = r - 1, \quad \eta = \theta, \quad g = \sin^2\theta, \quad \Omega = 3\sin^2\theta\cos\theta. \quad (52)$$

Here the characteristic velocity scale is taken to be $U_* = \alpha a_e \mu_1 (\mu_1 + \mu_2)^{-1}$, α is the shear factor.

By virtue of [8] and the results of Section 7 the integral diffusional flux, in case the diffusion coefficient is an arbitrary function of concentration and there is a sudden occurrence of reaction on the droplet surface, is determined from

$$I = I_\alpha = \frac{\sqrt{\pi}}{2} J(B, A) I_\alpha^0, \quad I_\alpha^0 = 4[3\pi Pe \operatorname{cth}(3t)]^{1/2}. \quad (53)$$

Equations (32), (36) and (53) determine the integral flux to the droplet surface for linear and hyperbolic dependence of the diffusion coefficient upon concentration (27) and (33).

9. UNSTEADY STATE DIFFUSION TO THE FILM SURFACE. PERIODIC REGIMES

Consider a convective diffusion to the surface of a film falling down a wall in the periodic regime. It is often the case that the hydrodynamic model of film motion leads to the following expression for the velocity components in equations (1) and (2) [3, 4, 6, 26]:

$$\Omega = \Omega(y), \quad y = \eta - vt, \quad g = 1\ (v > 0). \quad (54)$$

Here $\Omega(y) = \Omega(y + T_0)$ is a certain (arbitrary)'periodic function y with the interval T_0, the specific form of which is determined from solution of the respective hydrodynamic problem.

In a general case the variables (4) corresponding to (54) can be obtained from equations (11), (13) and (14). Then, distribution of concentration for the developed periodic regimes with a steady supply of reagent at the point of entry into the reactor, which corresponds to the boundary conditions $\eta = 0$, $c = \psi_\beta = b$; $\xi = 0$, $c = \varphi = a$(a, $b = \text{const.}$) is determined, for equations (1), (2) and (54), by the equality $c = c_\beta$, where the function c_β is defined in (23), (25), while the variables f and τ_β are specified by equations (11), (14). In the linear case, $\sigma = 1$, such problems were considered in [3, 4, 6].

Let us obtain some other exact time-periodic solutions of the problem (1), (2), (54). In terms of the new variables η and y the differential operator $\mathbf{L}$ (5), (54), which determines the form of the variables (4) and (8), acquires the form

$$\mathbf{L} = (\Omega - v)\frac{\partial}{\partial y} + \dot{\Omega}\frac{\partial}{\partial \eta} \quad (\Omega'_y = \Omega'_\eta). \quad (55)$$

Therefore, it is convenient to choose the variables (4), (8) in this case in the form

$$f(y) = \Omega(y) - v,$$

$$\omega = \eta - y - v\int_0^y f^{-1}\,\mathrm{d}y, \quad \tau_0(y) = \int_0^y f\,\mathrm{d}y. \quad (56)$$

When writing equations (56) it was assumed that $\Omega - \nu > 0$; τ_0 is a particular solution of the last equation in (8), (55). In order that the final form of the variable τ be chosen, let us avail ourselves of the remark made at the beginning of Section 8, i.e. that the general solution of equations (8) and (55) for τ has the form

$$\tau = \int_0^y f\,\mathrm{d}y + Z\left(\eta - y - \nu\int_0^y f^{-1}\,\mathrm{d}y\right). \quad (57)$$

Here $Z = Z(\omega)$ is an arbitrary function the specific form of which is determined from the condition of periodicity of the variable (57) in y (or, equivalently, in t). It follows from the condition $\tau(y, \eta) = \tau(y + T_0, \eta)$ that the function Z will generate the equation

$$Z(\omega) = Z[\omega - T_0(1 + \nu < f^{-1} >)] + T_0 < f >,$$

$$< h > \equiv \frac{1}{T_0}\int_0^{T_0} h\,\mathrm{d}y. \quad (58)$$

The solution of equation (58) is sought in the form

$$Z(\omega) = p\omega + q \ (p, q = \mathrm{const.}). \quad (59)$$

Substitution of (59) into (58) leads to the following value of the coefficient p

$$p = < f > (1 + \nu < f^{-1} >)^{-1}. \quad (60)$$

As would be expected, the coefficient q in this case is an arbitrary additive constant.

With regards to (57–60) we obtain for the variable τ

$$\tau = \int_0^y f\,\mathrm{d}y + \frac{< f >}{1 + \nu < f^{-1} >}\left(\eta - y - \nu\int_0^y f^{-1}\,\mathrm{d}y\right) + q$$

$$= \int_0^y f\,\mathrm{d}y + \frac{\nu < f >}{1 + \nu < f^{-1} >}\left(t - \int_0^y f^{-1}\,\mathrm{d}y\right) + q,$$

$$y = \eta - \nu t. \quad (61)$$

Now, selecting the parameter q so that the inequality $\tau > 0$ is valid, we obtain, for any function Ω, the one-parametric set of solutions of equations (1), (2) and (54), which satisfies the boundary conditions $\xi = 0$, $c = a$; $\xi = \infty$, $c = b$, in the form

$$c = Y\left(\frac{\xi f}{2\varepsilon\sqrt{\tau}}, a, b\right). \quad (62)$$

The variables f and τ in the above equation have been defined by equations (56) and (62), while the function $Y(x, a, b)$ is the solution of the problem (25).

By virtue of (24), (25), (56) and (61), the local diffusional flux to the film surface, corresponding to the solution (62), at $q = O(1)$ and $\eta \to \infty$ has the form

$$\eta \to \infty, \quad j = \frac{1}{2}Pe^{1/2}J(\sigma, a, b)$$

$$\times\left(\frac{1 + \nu < f^{-1} >}{< f >}\right)^{1/2} f(y)\eta^{-1/2}. \quad (63)$$

This expression shows that at $\eta \to \infty$ the local diffusional flux decreases inversely proportional to the square root of the distance from the entrance to the reactor, with the proportionality factor depending upon the mean values of f and f^{-1} in the oscillation period.

Equation (63) gives the expression for the mean local diffusional flux:

$$\eta \to \infty, \quad < j > = \frac{1}{2}Pe^{1/2}J(\sigma, a, b)$$

$$\times [< f > (1 + \nu < f^{-1} >)]^{1/2}\eta^{-1/2}. \quad (64)$$

It can be shown that at $\eta \to \infty$ these very equations, (63) and (64), define the principal term of the asymptote to the quantities j and $< j >$ obtained for the developed periodic mode of diffusion with a steady state supply of reacting substance at the boundary conditions $\xi = 0$, $c = a$; $\eta = 0$, $c = b$. The case of the linear problem, $\sigma = 1$, corresponds to $J = 2\pi^{-1/2}(b - a)$ in equations (63) and (64).

At $\Omega - \nu < 0$, the variable f is chosen in the form $f = \nu - \Omega$, while the variables ω and τ are obtained from equations (56) and (61) by substituting $-f$ for f.

10. UNSTEADY STATE DIFFUSION IN THE CASE OF ARBITRARY KINETICS OF SURFACE REACTION

Now consider the process of unsteady state convective diffusion to a droplet (bubble, liquid film) with the surface chemical reaction occurring on its surface with the final rate $k'F_*(c_s)$, where k' is the reaction rate constant, F_* is the "reaction law".

A corresponding boundary value problem is described by the linear equation for concentration in the bulk of the liquid $\Lambda(c, 1) = 0$ and by initial and boundary (at $\eta = 0$) conditions (3) and (41). Moreover, that the formulation of the problem be completed, it is necessary to add an equation for the surface concentration distribution

$$\xi = 0, \quad \frac{\partial c_s}{\partial t} + \frac{1}{\sqrt{g}}\Omega\frac{\partial c_s}{\partial \eta} = -k'F_*(c_s) + D\frac{\partial c}{\partial \xi}, \quad (65)$$

$$\xi = 0, \quad c = \Gamma c_s. \quad (66)$$

In writing equation (65) the term $D_s\Delta_\eta c_s$, which describes molecular surface diffusion, was ignored and it was assumed that there is phase equilibrium between the bulk and surface concentration (66); Γ is the Henry constant. For convenience, all the equations and initial and boundary conditions are presented in dimensional form which formally is equivalent to the replacement $\varepsilon = \sqrt{D}$ at $\sigma = 1$ in equations (1) and (2) and the variable ζ (4).

Turning to the variables (4), (11), (13) and (14), with regard for (65) and (66) and $\partial/\partial\tau = f^{-2}\mathbf{L}$, we obtain the following problem for concentration distribution

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial \zeta^2}; \ \tau = 0, \quad c = A \quad (67)$$

$$\zeta = 0, \quad f^2\frac{\partial c}{\partial \tau} = -F(c) + \Gamma\sqrt{D}f\frac{\partial c}{\partial \zeta};$$

$$F(c) \equiv k'\Gamma F_*(\Gamma^{-1}c).$$

Here the subscripts α and β, corresponding to adjacent regions located on each side of the curve $\omega = 0$ in the plane η, t (Fig. 2), on the variable τ and the parameter A are omitted.

The solution of the problem (67) is sought in the form

$$c(\omega,\zeta,\tau) = A - \frac{1}{\sqrt{\pi}}\int_0^\tau \frac{\Phi(\omega,\lambda)}{\sqrt{(\tau-\lambda)}}\exp\left\{-\frac{\zeta^2}{4(\tau-\lambda)}\right\}\mathrm{d}\lambda \tag{68}$$

Here c is the solution of equation (67) and it satisfies the initial condition at $\tau = 0$ for any limited kernel Φ; then the following limiting relations [27] hold:

$$\lim_{\zeta\to 0} c = C = A - \frac{1}{\sqrt{\pi}}\int_0^\tau \frac{\Phi(\omega,\lambda)\,\mathrm{d}\lambda}{\sqrt{(\tau-\lambda)}},$$

$$\lim_{\zeta\to 0}\frac{\partial c}{\partial\zeta} = \Phi(\omega,\tau). \tag{69}$$

Substituting equation (68) into the last boundary condition in (67) and taking into account the properties (69), we obtain a system of equations to determine the limiting concentration $C = c(\omega, 0, \tau) = c(t, 0, \eta)$ and the kernel Φ

$$f^2\frac{\partial C}{\partial\tau} = \Gamma\sqrt{D}f\Phi - F(C), C = A - \frac{1}{\sqrt{\pi}}\int_0^\tau \frac{\Phi\,\mathrm{d}\lambda}{\sqrt{(\tau-\lambda)}}$$

Solving the first equation of this system for Φ and substituting the respective expression into the second equation, we obtain the integro-differential equation for the function C:

$$C = A - \frac{1}{\Gamma\sqrt{(\pi D)}}\int_0^\tau\left[F(C) + f^2\frac{\partial C}{\partial\lambda}\right]\frac{\mathrm{d}\lambda}{f(\omega,\lambda)\sqrt{(\tau-\lambda)}}. \tag{70}$$

For a steady state flow the following equalities hold

$$\tau_\beta = \tau_\beta(\eta) \text{ and } f^2\,\partial/\partial\tau_\alpha = \partial/\partial t + f^2\,\partial/\partial\tau_\beta,$$

which makes it possible in Case (b) [which corresponds to $\tau = \tau_\alpha$ in (70)], on having replaced $x = \lambda + \tau_\beta[T(\omega)]$ in the integrand, to write equation (70) in the form

$$C = A - \frac{1}{\Gamma\sqrt{(\pi D)}}$$

$$\int_{\tau_\beta}^{\tau_\beta}(T(\omega))\left[F(C) + \frac{\partial C}{\partial t} + f_*^2\frac{\partial C}{\partial x}\right]\frac{\mathrm{d}x}{f_*(x)\sqrt{(\tau_\beta - x)}},$$

$$f_*(\tau_\beta) \equiv f[\eta(\tau_\beta)], \quad \tau_\beta = \tau_\beta(\eta). \tag{71}$$

Now, let us analyze one particular case of flow when equation (71) is integrated explicitly. To this end, consider a plane steady state flow of an ideal incompressible fluid in the region $\xi \geq 0$, $\eta \geq 0$ which is determined by the stream function

$$\Psi = \delta\xi\eta, \quad g = 1\ (\Omega = \delta\eta, \quad \delta = \text{const.}). \tag{72}$$

The new variables (4), (11), (13) and (14), entering into equation (71) and corresponding to the non-linear non-stationary problem (1), (2), (65), (66) and (72) have the form

$$\omega = -t + \delta^{-1}\ln\eta, \quad f = \delta\eta, \quad \tau_\beta = \delta\eta^2/2,$$

$$T(\omega) = \mathrm{e}^{\delta\omega}, \quad f_*(x) = (2\delta x)^{1/2} \tag{73}$$

By passing to a new integration variable $z = x\tau_\beta^{-1}$ in the integral (71), the solution of equations (71) and (73) is sought in the form $C = C(t)$. With regard for the equality $\tau_\beta(\eta)\ \tau_\beta^{-1}\ (T(\omega)) = \mathrm{e}^{-2\delta t}$, this yields the following ordinary differential equation for the limiting concentration:

$$C = A - \frac{1}{\Gamma\sqrt{(2\pi\delta D)}}$$

$$\times\left[\pi - B\left(\mathrm{e}^{-2\delta t},\frac{1}{2},\frac{1}{2}\right)\right]\left(\frac{\mathrm{d}C}{\mathrm{d}t} + F(C)\right). \tag{74}$$

Equation (74) is easily integrated explicitly in the case of the 1st-order surface reaction $F(C) = kC$; here, just as in the general case, the following natural condition $C(0) = A$ should be chosen as an initial condition for the surface concentration.

It should be noted that even though equation (74) for the limiting concentration has been obtained in the diffusion boundary layer approximation (1) and (2), yet it is an exact solution of the complete convective diffusion equation

$$\partial c/\partial t + (\mathbf{v}\nabla)c = D\Delta c; \ v_\xi = -\partial\Psi/\partial\eta,$$

$$v_\eta = \partial\Psi/\partial\xi. \tag{75}$$

This can be verified directly by substituting the expression for the stream function (72) into (75) with regard for the fact that the solution of the problem (72), (75) is independent of the transverse coordinate η, i.e. $c = c(t, \xi)$.

Equation (74) also describes the dependence of the limiting surface concentration upon time at the forward stagnation point of a spherical droplet $\theta = 0$ in a steady state translational Stokesian flow; in this case the parameter δ in equation (74) should be set equal to $\delta = U_0\,\mu_1[2a_e(\mu_1 + \mu_2)]^{-1}$.

For a stationary nonlinear boundary condition on the droplet surface the solution of the problem

$$\Lambda(c, 1) = 0;\ \xi = 0, \quad \partial c/\partial\xi = R_s(c)$$

subject to the initial and boundary (at $\eta = 0$) conditions (3) and (41) is reduced to the solution of the following integral equation for the limiting concentration [11]

$$C = A - \frac{\varepsilon}{\sqrt{\pi}}\int_0^\tau \frac{R_s(C)\,\mathrm{d}\lambda}{f(\omega,\tau)\sqrt{(\tau-\lambda)}}.$$

It should be noted that equation (68) at $\Phi = \varepsilon f^{-1}H$ gives the solution of the linear boundary value problem $\Lambda(c, 1) = 0$ with the initial and boundary (at $\eta = 0$) conditions (3), (41) for the case when the local flow on the reacting surface is prescribed in the form

$$\xi = 0, \quad \partial c/\partial \xi = H(t,\eta),$$

where H is the known function.

11. STATIONARY CASE. CERTAIN CORRELATIONS

For the stationary problems (1) and (2) $[\partial/\partial t = 0,\ \Omega = \Omega(\eta),\ g = g(\eta)]$ subject to the boundary conditions

$$\eta = 0, \quad c = A; \quad \xi = 0, \quad c = B \ (A, B = \text{const.}) \tag{76}$$

it follows from the results of Section 7 that the integral diffusional flux to the droplet surface is determined by

$$I = \frac{\sqrt{\pi}}{2} J(B,A) I^0, \quad I^0 = \frac{4}{\varepsilon}\left(\pi \int_0^{\eta^+} \sqrt{g\Omega}\,\mathrm{d}\eta\right)^{1/2} \tag{77}$$

Here I^0 is the integral diffusional flux corresponding to the linear ($\sigma = 1$) stationary problem (1) and (2) subject to the boundary conditions (76) at $A = 1, B = 0$; the formula for I^0 is presented for an axisymmetric case. In Case (b) the first boundary condition (76) should be replaced by the condition at infinity $\xi \to \infty$, $c \to A$.

In the case of an arbitrary dependence of the diffusion coefficient on concentration, equations (77) can correlate the results of the works [14, 28] obtained for a chain of droplets (bubbles). In particular, for a chain of droplets of equal radius located at a (non-dimensional) distance l from each other on the axis of the translational Stokesian flow (Fig. 7), provided that

$$O(1) < l < O(\varepsilon^{-1}), \quad \sigma_1 \leq \sigma(c) \leq \sigma_2; \quad 0 < \sigma_1 \leq \sigma_2 < \infty,$$

the following formulae are valid for the integral diffusional fluxes at the droplet surfaces

$$I_m = I_1[m^{1/2} - (m-1)^{1/2}] \ (m = 1,2,\ldots). \tag{78}$$

Here m is the ordinal number of a drop; the numbering starts as shown in Fig. 7.

For a stationary problem similar results can also be obtained for a chain of solid spheres in a Stokesian flow. The stream function in the vicinity of surfaces will be quadratic in ξ and will have the form $\Psi = \xi^2\Omega$. By using the same sort of reasoning as in [29, 30], we obtain the following formulae for the integral diffusional fluxes:

$$I_m = I_1[m^{2/3} - (m-1)^{2/3}] \ [O(1) < l < O(Pe^{1/3})]. \tag{79}$$

Equations (78) and (79) show that in the general case of an arbitrary dependence of diffusion coefficient upon concentration the ratio, I_m/I_1, of the integral diffusional fluxes to droplets or solid particles of the chain does not depend on the diffusion coefficient $\sigma(c)$. Here, just as in the linear case, the integral mass transfer in the chain is strongly retarded by interacting diffusional wakes and boundary layers of particles in the chain.

It should be emphasized in conclusion that the procedure of introduction of new variables (4) and (8) can be applied to analyze similar, but more complex, boundary value problems (and the systems of equations), to those presented by equations (1) and (2). Thus, it can be used for the study of a number of processes of simultaneous heat and mass transfer. Moreover, in the case of a compressible fluid one can use a more general (than (4) and (8)) method of introduction of new independent variables which makes it possible to analyze the corresponding problems of the unsteady state boundary layer. Namely, if the fluid velocity components at the particle surface ($\xi \to 0$) can be approximated as

$$v_\xi = \xi V(t,\eta), \quad v_\eta = \Omega(t,\eta), \tag{80}$$

then, to simplify equations (1) and (80) (it is supposed here for simplicity that the metric coefficients have been chosen conventionally $g_{\xi\xi} = g_{\eta\eta} = 1$), one should introduce the new variables (4) and require that the functions ω, f, τ would satisfy the equations

$$\mathbf{L}^*\omega = 0, \quad \mathbf{L}^* f = -Vf, \quad \mathbf{L}^*\tau = f^2; \ \mathbf{L}^* \equiv \partial/\partial\tau + \Omega\partial/\partial\eta. \tag{81}$$

It is seen that the systems (8) and (81) are quite similar in structure. The function $f = f(t, \eta)$ should therefore be determined from

$$f = \exp\left\{-\int_{\eta_*}^{\eta}\left(\frac{V}{\Omega}\right)_{(\omega)} \mathrm{d}\eta\right\}, \tag{82}$$

while the variables ω and τ are prescribed by equations (10), (11), (13), (14) and (82) at $g \equiv 1$. Equations (1) and (80), on employing substitutions (4) and (81), are reduced to equation (9).

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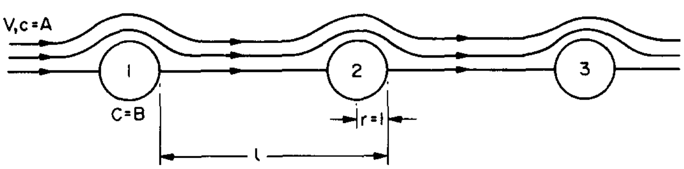

FIG. 7. Scheme of translational flow past a chain of spherical particles or droplets.

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RESOLUTION DE QUELQUES PROBLEMES NON-LINEAIRES DE COUCHE LIMITE (THERMIQUE) INSTATIONNAIRE ET CONTROLEE PAR LA DIFFUSION

Résumé—On suggère une méthode analytique exacte pour la résolution d'une large classe de problèmes de couche limite instationnaire qui décrivent les mécanismes de transfert convectif de chaleur et de masse des particules dans un fluide incompressible et visqueux avec des gouttelettes (bulles). La méthode se révèle aussi adaptée pour les mécanismes de transfert au voisinage de l'interface liquide-fluide (comme pour les films liquides). La méthode est basée sur l'introduction de quatre nouvelles variables indépendantes spécifiées par un système d'équations aux dérivées partielles du premier orde qui admet une solution générale du problème linéaire avec des conditions initiales et aux limites arbitraires. De plus, il est possible d'obtenir une large classe de solutions de problèmes non-linéaires dans le cas où le coefficient de diffusion (diffusivité thermique) est une fonction de la concentration (température).

Une solution générale est obtenue pour un problème linéaire, similaire et instationnaire connecté à une réaction chimique du premier ordre dans le fluide. On étudie la diffusion à la particule pour une vitesse arbitraire de réaction sur sa surface.

On montre que la solution de l'équation de couche limite instationnaire peut avoir une forte discontinuité qui se déplace à une vitesse terminale.

EIN VERFAHREN FÜR DIE LÖSUNG VON NICHTLINEAREN RANDWERTPROBLEMEN EINER NICHTSTATIONÄREN DIFFUSIONSBESTIMMTEN (THERMISCHEN) GRENZSCHICHT

Zusammenfassung—Ein exaktes analytisches Verfahren zur Lösung einer weiten Klasse von instationären Grenzschichtproblemen wird vorgeschlagen, welche die Vorgänge des konvektiven Stoff- und Wärmeübergangs von Teilchen in einer idealen Flüssigkeit und Tropfen (Blasen) in einer viskosen inkompressiblen Flüssigkeit beschreiben. Das Verfahren ist auch auf die Berechnung von Übertragungsvorgängen anwendbar, die in der Umgebung einer Gas-Flüssigkeitsgrenze (d.h. in Flüssigkeitsfilmen) auftreten. Die Methode basiert auf der Verwendung von vier neuen unabhängigen Variablen, die durch ein System partieller Differentialgleichungen erster Ordnung bestimmt sind, wodurch eine allgemeine Lösung des linearen Problems bei beliebigen Anfangs- und Randbedingungen ermöglicht wird. Zusätzlich bietet sie die Möglichkeit, eine weite Klasse nichttrivialer (und nicht ähnlicher) Lösungen von entsprechenden nichtlinearen Randwertproblemen für den Fall zu erhalten, daß der Diffusionskoeffizient (die Temperaturleitfähigkeit) eine Funktion der Konzentration (Temperatur) ist.

Eine allgemeine Lösung erhält man für ein ähnliches lineares instationäres Problem, das mit einer chemischen Reaktion erster Ordnung im Fluid verbunden ist. Dabei wird der Vorgang der konvektiven Diffusion an einem Teilchen bei beliebiger chemischer Reaktionsgeschwindigkeit an dessen Oberfläche untersucht. Es wird gezeigt, daß die Lösung der instationären Grenzschtgleichung eine starke Unstetigkeit aufweisen kann, die sich bei der Endgeschwindigkeit verlagert.

МЕТОД РЕШЕНИЯ НЕКОТОРЫХ НЕЛИНЕЙНЫХ КРАЕВЫХ ЗАДАЧ НЕСТАЦИОНАРНОГО ДИФФУЗИОННОГО (ТЕПЛОВОГО) ПОГРАНИЧНОГО СЛОЯ

Аннотация — В работе предлагается точный аналитический метод решения широкого класса нестационарных погранслойных задач, описывающих процессы конвективного массо- и теплообмена частиц в идеальной и капель (пузырей) в вязкой несжимаемой жидкости. Указанная процедура пригодна также для анализа процессов переноса, происходящих в окрестности поверхности раздела типа жидкость–жидкость (газ) (например, в пленках жидкости). Метод основан на введении новых четырех независимых переменных, удовлетворяющих системе уравнений в частных производных первого порядка, и позволяет получить общее решение линейной задачи в случае произвольных начальных и граничных условий. Кроме того он дает возможность получить широкий класс нетривиальных (и неавтомодельных) решений соответствующих нелинейных краевых задач в случае, когда коэффициент диффузии (температуропроводности) зависит от концентрации (температуры).

Получено общее решение аналогичной линейной нестационарной задачи при протекании в жидкости объемной химической реакции первого порядка. Исследуется процесс конвективной диффузии к поверхности частицы при произвольной кинетике поверхностной химической реакции.

Показано, что решение нестационарного уравнения пограничного слоя может иметь сильный разрыв, который перемещается с конечной скоростью. Указано обобщение метода для аналогичных задач нестационарного массо- и теплообмена в сжимаемой жидкости, а также при наличии вдува или отсоса жидкости на реагирующей поверхности.

Предложенный метод используется для решения ряда конкретных задач. Подробно исследуется случай линейной и гиперболической зависимости коэффициента диффузии от концентрации.